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Pricing Equity-Linked Life Insurance Contracts with Minimum Interest Rate Guarantee in Partial Equilibrium Framework

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Zusammenfassung

In der vorliegenden Studie wird das Pricing aktiengebundener Lebensversicherungen mit Mindestgarantiezins in einem Gleichgewichtsmodell untersucht. Dazu werden Modelle zur Berechnung der Default-Option und der Ausfallwahrscheinlichkeiten eingeführt. Die Modelle binden Angebot- und Nachfrageüberlegungen, stochastische Zinssätze, stochastische Kapitalrenditen und Sterblichkeitsraten ein. Dabei basiert die betrachtete Zinsstruktur auf dem Cox, Ingersoll, Ross (1985) Modell. Der Einfluss der Startwerte der Default-Option auf die endgültigen Werte der Default-Option und der Ausfallwahrscheinlichkeiten wird numerisch analysiert, wobei die verwendeten numerischen Methoden eine hohe Konvergenzgeschwindigkeit und Robustheit im Hinblick auf die Startwerte der Defaultoption aufweisen. Es wird gezeigt, dass höhere garantierte Mindestrenditen zu höheren Ausfallwahrscheinlichkeiten, zu niedrigeren optimalen Kapitalwerten und veränderten Schadenzahlungen führen. Deshalb müssen bei der Ermittlung der Mindestgarantie sowohl Risiko und Gewinn, als auch Vor- und Nachteile vorhandener Alternativen berücksichtigt werden.

Schlagwörter: Ausfallwahrscheinlichkeit; garantierte Mindestrendite; aktiengebundene Lebensversicherung

Abstract

This paper examines the pricing of equity-linked life insurance including a minimum interest rate guarantee in a partial equilibrium framework. The models for calculating default option values and default probability are established. Those models integrate supply and demand considerations, stochastic interest rates, stochastic investment return, and mortality rates. The term structure of interest rates is based on the Cox, Ingersoll, Ross (1985) model. The paper analyses numerically the influence of initial values of default option on final default option values and default probabilities. Our numerical methodologies have high convergence speeds and the convergence process is robust with respect to initial values of default option. Our results indicate that increased minimum guaranteed return rate will result in increased default probability, decreased optimal net present value and changed values of claim payment. Therefore, the process of setting the minimum interest rate guarantee must take into account risk and benefit, as well as advantages and disadvantages of available alternatives.

Keywords: Default Probability; Minimum return guarantee; Equity-Linked life insurance

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1 Introduction

All business firms rely upon a variety of forms of financing. However, insurance companies like, similarly to most other financial institutions (e.g., banks) are different from business firms because they create explicit liabilities whenever they sell their products. Indeed, its policyholders hold most of a typical insurer's liabilities. An insurer's success depends not only on charging adequate rates to cover costs, but also on being a negligible default risk to policyholders. Therefore, policyholders are wiling to pay for policies with comfortably low risk, and insurers must control default risks in order to sustain a competitive position in the marketplace.

Mao, Carson, Ostaszewski and Luo (2004) extend option pricing models (Black and Scholes (1973), Cummins (1991), Grosen and Jørgensen (2000)) to measure default put option value, and to pricing of term life policies. Their approach incorporates stochastic interest rate and correlation between two stochastic processes of interest rate and accumulated investment. In this work, we extend that approach to measure default option value and default probabilities within the context of an equity-linked life insurance product and determine optimal pricing for equity linked insurance contracts. There are some papers that discuss the default risk and insolvency put option. Cummins and Danzon (1997) developed a two period pricing model subject to default risk. Demand for insurance is assumed imperfectly price elastic. Babbel et al. (2002) discussed fair value of liabilities based on financial economics. He points out that the fair value of premium (or liability) should be discounting liability cash flows at treasury rates minus the value of insolvency put options, as there must be some accounting for the default risks. In the established model, we use their analysis results and consider the affect of the value of insolvency put option on the price of equity linked life insurance contracts. We also consider the solvency constraint. The contribution of this paper is the application of the Mao et al model to equity-linked term life insurance contracts, and the analysis of the effects of different parameters on insurance prices in this partial equilibrium model. A life insurance policy is said to be the equity-linked when the benefits are based on the market value of a specified stock investment portfolio. Based on the characteristic of U.S. products, the minimum guarantee indicates guaranteed minimum death benefit with interest guarantees. There have been several papers in which pricing of the equity-linked life insurance was discussed. Brennan and Schwarts (1976, 1979) pioneered this type of analysis focusing on the case of an endowment policy. Bacinello (1993) extended Brennan and Schwartz's approach to the case of endogenous minimum guarantees. Milevsky and Posner (2001) discussed variable annuities insurance with guaranteed interest rate and guaranteed death claim payment. Bacinello and Persson (2002) proposed the design and pricing of equity-linked life insurance under stochastic interest rates, which uses the term structure of interest rates proposed by Heath, Jarrow and Morton (1996). The advantage of the method proposed in our work, when compared with existing approaches, is that relevant options are easy to price and may be hedged either by long positions in the appropriate asset portfolio or by European call options on the same portfolio.

Von Neumann and Morgenstern (1944) pointed out that the when economic decision makers interact, both though decisions and behaviors, they form relationship of influencing each other. Therefore, an economic principal must reflect on his/her counterpart when making decisions. In line with that idea, we establish models of measuring credit risk and pricing equity-linked life insurance contract by balancing risks and revenue, at the same time, considering influences of default risk and claim payment paid by insurers (including investment revenue) on prices consumers are willing to pay for policies. Additionally, we propose a rational principle for selecting minimum guaranteed return rate.

2 Pricing and Measurement Models of Default Risk

2.1 Assumptions Underlying the Measurement Models

In this section, we discuss the assumptions underlying the development of pricing and measurement models used in this study.

The insurer sells only life insurance. We consider single-premium equity-linked life insurance policies with term Y. The contract is a contingent-claim affected by both mortality and financial risk. Stochastic interest rates are used as discount rates and are derived by a continuous-time stochastic process. Mortality is also modeled as a random phenomenon (this contrasts with Giaccotto, 1986; and Panjer and Bellhouse, 1980). Our model treats the insurance policyholder as an investor buying a financial asset. The insurer raises funds from policyholders, invests the funds, and pays benefits/claims that may depend on investment income.

The insurance firm is assumed to have market power and thus be able to vary its premium volume by varying price (i.e., we do not assume perfectly competitive insurance markets, but let us note that a decreasing demand function also applies to the aggregation of all companies in a competitive industry, so our assumption is not too severely restrictive). Capital markets are assumed to be perfectly competitive, frictionless, and free of arbitrage opportunities. All consumers purchase the same unit of insurance coverage, and their demand is a function of price, the claim payment including investment return, and default risk. Moreover, all policyholders are assumed to be rational and non-satiated, and to share the same information.

Assume that the insurance firm considers a price for policies that is a function of quantities of policies, insolvency risks (financial quality) and claim payment:

$$PP\left(n, b\left(n, r_g\right), \pi\left(n, r_g\right)\right),$$

where $n =$ quantity of insurance sold; $\pi =$ expected present value of claim payment for each exposure unit; $Y =$ maturity time of insurance contracts; $r_g =$ minimum guarantee return rate; and $b =$ the value of the default option—the current value of the insurance firm's option to default if liabilities exceed assets at the claim payment date; $QD\left(n, r_g\right) =$ default probability—the probability that liabilities exceed assets at the claim payment date. The default option value and default probability are inversely related to the price and liability. The following notation is used:

$b\left(n, r_g\right)$: the value of the default put option
QD : default probability
n : quantity of insurance sold for single premium equity–linked life insurance policy
r_g : minimum guaranteed return rate
$PP(n, b(n, r_g), \pi(n, r_g))$: life insurance single premium based on market price (i.e., not necessarily single benefit premium or any form of gross premium)
$ENPV(n, r_g)$: expected net present value of the life insurance policy under consideration
p : non-claim payment expense percentage of claim payment

h: percentage of total investment income that can be used to pay benefit to the insured
$X(t)$: benefit payable (or claim payment, in general, if the insured event is something other than death) upon death at time *t*
$T(x)$: future lifetime of an insured, the said insured assumed to be *x* year old
Y: the maturity time of the insurance contract
$f_x(t)$: the probability density function of *T,* also dependent on the age of issue *x*
p_{x+t} : the probability that insured aged *x* survives from age *x* + *t* to age *x* + *t* + 1
$q_{x+t} = 1 - p_{x+t}$
${}_t p_x = \Pr\left(T(x) > t\right)$
${}_t q_x = 1 - {}_t p_x$
π: expected present value of claim payment for each exposure unit
AA: constant coefficient of the demand function (assumed linear) for single premium equity-linked life policy
B, G, F: coefficients of the demand function for single premium equity-linked life policy corresponding to quantity of demand, insolvency risk, and expected present value of claim payment for each exposure unit respectively
μ: long run equilibrium interest level (assumed independent of *t*)
σ_1: standard deviation of cumulative investment in the predetermined investment portfolio of the policy, which we will refer to as a mutual fund (assumed independent of *t*)
r: short run interest rate
σ: standard deviation of interest rate (assumed independent of *t*)
κ: the speed of adjustment in the mean reverting process
$v(t) = \exp\left(-\int_0^t r_u du\right)$ = the discount function; it represents the time zero value of one unit of account at time t discounted using interest rates determined by a stochastic process

Table 1: Notation

2.2 Pricing Model of Default Option Value

Suppose that the initial asset D_0 is the premium income of the insurance firm so that

$$D_0 = PP\left(n, b\left(n, r_g\right), \pi\left(n, r_g\right)\right)n.$$

The funds are invested in a mutual fund. Let us denote by D_t the market value at time t of the accumulated investment in the mutual fund and by h the percentage of investment income, which can be used to pay benefit to the insured. We write

$$PUT_t = \max(X_t - D_t, 0) \qquad (1)$$

for the value at time t of a put option on the accumulated investment with exercise price X_t, where X_t is the cash flow of liability at time t and

$$X_t = \left(\max\left(D_t h, GUE_t\right)\right)\left(1+p\right) = \left(D_t h + \max\left(GUE_t - D_t h, 0\right)\right)\left(1+p\right), \qquad (2)$$

where GUE_t is minimum determined guaranteed account value as related to time, $GUE_t = n_t q_x e^{r_g t}$, and r_g is the minimum guaranteed return rate.

We assume that the benefit for the insured for each contract is one dollar (or a monetary unit in general). We assume D_t is described by the following stochastic differential equation under the equivalent martingale measure:

$$dD_t = rD_t dt + \sigma_1 D_t dw^1,$$

where r satisfies stochastic differential equation

$$dr = \kappa\left(\mu - r\right)dt + \sigma\sqrt{r}dw^2,$$

and dw^1, dw^2 are two Wiener processes, possibly correlated, and $\rho_{1,2}$ is their instantaneous correlation coefficient. This model is based on the approach of Cox et al (1985). The model is mean-reverting in the sense that the rate of return tends to revert to the long-term average μ, with volatility of the rate proportional to the square root of the rate. Mean reversion is commonly assumed in models of interest rates, or returns on bond portfolios, because very low interest rates are associated with severe recessions, while very high interest rates are associated with high inflation, and both of these extremes are typically counteracted with economic policy. But returns of stocks also may be modeled as mean-reverting. Fama and French (1988) produced a seminal work on mean reversion in stock returns. We believe the model proposed to adequately represent investment portfolios of U.S. insurers, which tend to invest mostly in bonds, but further work research may be needed for portfolios for which mean-reversion does not apply.

Based on the definition of insolvency put option and equation (1), the current value of insolvency put option is equal to the accumulated discounting value of put options with exercise price

$$X_t = \left(D_t h + \max\left(GUE_t - D_t h, 0\right)\right)\left(1 + p\right),$$

for $0 < t < Y$, that is

$$b = b(n, r_g) = E^Q\left(\int_0^Y v(t)\, dPUT_t\right) =$$
$$= E^Q\left(\int_0^Y v(t) d\max\left(X_t - D_t, 0\right)\right) = E^Q\left(\int_0^Y A(t) e^{-B(t)r} d\max\left(X_t - D_t, 0\right)\right), \quad (3)$$

where

$$A(t) = \left(\frac{2\gamma e^{\frac{1}{2}(\kappa+\gamma)t}}{g(t)}\right)^{\frac{2\kappa\mu}{\sigma^2}}$$
$$B(t) = \frac{2\left(e^{\gamma t} - 1\right)}{g(t)}$$
$$g(t) = 2\gamma + \left(\kappa + \gamma\right)\left(e^{\gamma t} - 1\right)$$
$$\gamma = \sqrt{\kappa^2 + 2\sigma^2}$$

h indicates the percentage of total investment income that can be used to pay benefit to the insured and $E^Q[\bullet]$ denotes the expected value under the equivalent martingale measure. By using numerical approximation algorithms, we obtain the value of b. For a proof of existence of differential $d\max\left(X_t - D_t, 0\right)$, and a proof of continuity of function b please see the Appendix.

2.3 Measurement of Default Probability

We assume probability of default (default probability is similar to the "expected default frequency (EDF)" used by Moody's-KMV (Ong, 1999) is QD, and default event can occur before the maturity time. Then QD can be expressed as

$$
\begin{aligned}
QD(n,r_g) &= \Pr\left(D_Y < X_Y\right) = \Pr\left(\underset{t\le Y}{D_t} < \left(D_t h + \max\left(GUE_t - D_t h, 0\right)\right)\left(1+p\right)\right) = \\
&= \Pr\left(\underset{t\le Y}{D_t} < \frac{\max\left(GUE_t - D_t h, 0\right)\left(1+p\right)}{1-h\left(1+p\right)}\right),
\end{aligned} \tag{4}
$$

From equations of (3) and (4), we can know that the default boundary (default boundary is a level of asset value, possibly time dependent, such that the firm will default on its debt if the asset values falls below that level) is a function of time, policy number mortality rate and minimum guarantee return rate, and the default boundary is

$$
BARR_t = GUE_t \frac{1+p}{1-h\left(1+p\right)} = {}_t q_x n e^{r_g t} \frac{1+p}{1-h\left(1+p\right)}. \tag{5}
$$

Let the time horizon *Y* be divided into *s* time intervals of constant length. It can be shown that

$$
GUE_Y = \sum_{i=1}^{s} q_{x+\Delta t_i} n e^{r_g t_i} \tag{6}
$$

where $\Delta t_i = t_i - t_{i-1}$. Since

$$
\begin{aligned}
r_{t_i} - r_{t_{i-1}} &= \kappa(\mu - r_{t_{i-1}})\Delta t_i + \sigma\sqrt{r_{t_{i-1}}}\,\varepsilon_1\sqrt{\Delta t_i} \\
r_{t_i} &= r_{t_{i-1}} + \kappa(\mu - r_{t_{i-1}})\Delta t_i + \sigma\sqrt{r_{t_{i-1}}}\,\varepsilon_1\sqrt{\Delta t_i}, i = 1,2,\ldots,s,
\end{aligned} \tag{7}
$$

we have

$$
\begin{aligned}
D_{t_i} &= D_{t_{i-1}} + r_{t_{i-1}} D_{t_{i-1}} \Delta t + \sigma_1 D_{t_{i-1}} \varepsilon_2 \sqrt{\Delta t} = \\
&= D_0\left(\sum_{j=0}^{i-1}\left(1 + r_{t_i}\Delta t + \sigma_1\varepsilon_2\sqrt{\Delta t}\right)\right), \quad i = 1,2,\ldots s,
\end{aligned} \tag{8}
$$

where $\varepsilon_1 = \phi_1$, $\varepsilon_2 = \phi_1\rho_{1,2} + \phi_2\sqrt{1-\rho_{1,2}}$, and ϕ_1, ϕ_2 are independent standard normal random variables. We also have

$$
D_Y = D_0 \sum_{j=0}^{s-1}\left(1 + r_{t_i}\Delta t + \sigma_1\varepsilon_2\sqrt{\Delta t}\right) \tag{9}
$$

When parameters of $r_g, \mu, \kappa, \sigma, \sigma_1, Y, x, \rho_{1,2}$, the initial value of r_0, and the function of demand are given, by using numerical approximation, we can get the approximate solutions for default value b and default probability QD.

3 Optimization of Default Optimization of Default Boundary and Price of Insurance Contract

3.1 Objective Function

Since the default boundary is a function of time, policy number, and minimum guarantee return rate, we can establish objective function of maximizing expected net present value of profit, subject to the constrain of default probability less than or equal to a constant, which is assumed to be set endogenously by insurers or exogenously by insurance regulators, in order to find the optimum number of policies, minimum guarantee return rate, price and optimum default boundary. The optimization problem is to maximize returns(Revenues-expected claims+ value of default put option)while not allowing large default risks. We propose to quantify it as:

$$\text{Max } ENPV(n,r_g) = PP\Big(n, b\left(n,r_g\right), \pi\left(n,r_g\right)\ \Big)n - \pi n\left(1+p\right) + b, \tag{10}$$

subject to

(1) $PP\Big(n, b\left(n,r_g\right), \pi\left(n,r_g\right)\ \Big) > 0$

(2) $QD = QD\left(n,r_g\right) = \Pr\left(D_Y < X_Y\right) = \Pr\left(\underset{t\le Y}{D_t} < \dfrac{\max\left(GUE_t - D_t h, 0\right)\left(1+p\right)}{1-h\left(1+p\right)}\right) \le CON$

where CON is a constant. The minimum return guarantee means that if the investment income earned by the insured's investments is greater than minimum guarantee return, the insurer will pay the investment income, otherwise, it will pay minimum guarantee return. Based on this, cumulative present value of claim payment π can be expressed as cumulative present value of call options with exercise prices $GUE_t, 0 < t \le Y$, i.e.,

$$\begin{aligned}
\pi = \pi\left(n,r_g\right) &= \frac{1}{n}\int_0^Y E\left(\exp\left(-\int_0^t r_u du\right)X_t\right)dt \\
&= \frac{1}{n}\int_0^Y E\left(v(t)\left(D_t h + \max\left(GUE_t - D_t h, 0\right)\right)\right)dt \\
&= \frac{1}{n}\int_0^Y A\left(t\right)e^{-B(t)r}\left(D_t h + \max\left(GUE_t - D_t h, 0\right)\right)dt
\end{aligned} \tag{11}$$

$$A(t)=\left(\frac{2\gamma e^{\frac{(\kappa+\gamma)t}{2}}}{g(t)}\right)^{\frac{2\kappa\mu}{3\sigma^2}},$$

$$B(t)=\frac{2\left(e^{\gamma t}-1\right)}{g(t)},$$

$$g(t)=2\gamma+(\kappa+\gamma)\left(e^{\gamma t}-1\right),$$

$$\gamma=\sqrt{\kappa^2+2\sigma^2}.$$

$v(t)$ is the discount function and p is the non-claim payment expense percentage of claim payment.

From (10) we know that when consumer is sensitive to credit risk of insurer, the values of default put option have both positive and negative effects on expected net present values of profit ($ENPV$). On one hand, increasing the default put option value will decrease the value of liability and increase $ENPV$. On the other hand, increasing the default put option value will cause policyholders to pay a lower price and result in the decrease of $ENPV$. Therefore, the effect of default put option value on $ENPV$ depends on net results of these two effects.

3.2. Monte Carlo simulation and optimization

Let us illustrate our methodology with an example. Consider the demand function

$$PP(n,b(n,r_g),\pi(n,r_g))=AA-Bn-Gb+F\pi. \qquad (12)$$

In this function, the price is not only a function of market demand, but also a function of default risk and claim payment paid by insurers. Mao et al. (2004) provide justification for this form of demand. The difference between this work and Mao et al. (2004) is that in the present article, the insurance product is assumed to be equity-linked, so we consider the factor of minimum guarantee return rate that is supposed to be positively related to claim payment and also positively related to the price of insurance policies. The rationality of this assumption is clear: the greater the minimum guaranteed return rate, the greater investment total return obtained by consumers, and therefore, the greater the price. Since

$$D_0=\left(AA-Bn-Gb+F\pi\right)n, \qquad (13)$$

we have

$$D_Y = D_0 \sum_{i=0}^{s-1} \left(1 + r_{t_i} \Delta t + \sigma_1 \varepsilon_2 \sqrt{\Delta t}\right) = \\ = \left(AA - Bn - Gb + F\pi\right) n \sum_{j=0}^{s-1} \left(1 + r_{t_i} \Delta t + \sigma_1 \varepsilon_2 \sqrt{\Delta t}\right), \tag{14}$$

and

$$b = E^Q \left(\sum_{i=1}^{s} A(t_i) e^{-B(t_i) r_t} \Delta PUT_{t_i} \right) \\ = E^Q \left(\sum_{i=1}^{s} A\left(t_i\right) e^{-B\left(t_i\right) r_t} \left(\max\left(\left(X_{t_i} - D_{t_i}\right), 0\right) - \max\left(\left(X_{t_{i-1}} - D_{t_{i-1}}\right), 0\right)\right)\right). \tag{15}$$

We combine (8), (14), and (15) to obtain

$$b = E^Q \left(\sum_{i=1}^{s} A\left(t_i\right) e^{-B(t_i) r} \max\left(X_t - \left(AA - Bn - Gb + F\pi\right) n \sum_{j=0}^{i-1} \left(1 + r_{t_j} \Delta t + \sigma_1 \varepsilon_2 \sqrt{\Delta t}\right), 0 \right) - \\ - \max\left(\left(X_{t-1} - \left(AA - Bn - Gb + F\pi\right) n \sum_{j=0}^{i-2} \left(1 + r_{t_{j-1}} \Delta t + \sigma_1 \varepsilon_2 \sqrt{\Delta t}\right) \right), 0 \right) \right) \tag{16}$$

and equation (4) becomes

$$QD = \Pr\left(D_Y < X_Y\right) = \\ = \Pr \left(\underset{t_i \le T, i=1,2\cdots s}{D_{t_i}} < [\frac{\max\left({}_{t_i} q_x n e^{r_g t_i}, h\left(AA - Bn - Gb + F\pi\right) n \sum_{j=0}^{i-1} \left(1 + r_{t_j} \Delta t + \sigma_1 \varepsilon_2 \sqrt{\Delta t}\right) \right)}{1 - h\left(1+p\right)} - \right. \\ \left. - \frac{\max\left({}_{t_{i-1}} q_x n e^{r_g t_{i-1}}, h\left(AA - Bn - Gb + F\pi\right) n \sum_{j=0}^{i-2} \left(1 + r_{t_{j-1}} \Delta t + \sigma_1 \varepsilon_2 \sqrt{\Delta t}\right) \right)}{1 - h\left(1+p\right)} \right) \left(1+p\right) \tag{17}$$

where

$$\pi = \frac{1}{n}E^{Q}\left(\sum_{i=1}^{s} A(t_i)e^{-B(t_i)r}\max\left({}_{t_i}q_x ne^{r_g t_i}, h(AA - Bn - Gb + F\pi)n\sum_{j=0}^{i-1}\left(1 + r_{t_j}\Delta t + \sigma_1\varepsilon_2\sqrt{\Delta t}\right)\right) - \right.$$
$$\left. - \max\left({}_{t_{i-1}}q_x ne^{r_g t_{i-1}}, h(AA - Bn - Gb + F\pi)n\sum_{j=0}^{i-2}\left(1 + r_{t_{j-1}}\Delta t + \sigma_1\varepsilon_2\sqrt{\Delta t}\right)\right)\right) \tag{18}$$

and

$$X_t = \max\left({}_{t_i}q_x ne^{r_g t_i}, h(AA - Bn - Gb + F\pi)n\sum_{j=0}^{i-1}\left(1 + r_{t_j}\Delta t + \sigma_1\varepsilon_2\sqrt{\Delta t}\right)\right)(1+p)$$
$$- \max\left({}_{t_{i-1}}q_x ne^{r_g t_{i-1}}, h(AA - Bn - Gb + F\pi)n\sum_{j=0}^{i-2}\left(1 + r_{t_{j-1}}\Delta t + \sigma_1\varepsilon_2\sqrt{\Delta t}\right)\right)(1+p) \tag{19}$$

Parameters	Values of parameters	Parameters	Values of parameters
AA	0.075	q_{51}	0.00720
B	2×10^{6}	q_{52}	0.00784
G	0.0004	q_{53}	0.00857
r_0	0.07	Y	5
F	0.1	κ	0.24
p	0.2	σ	0.12
μ	0.05	ρ_{12}	0.50
σ_1	0.17	CON	0.05
q_{49}	0.00612	h	0.60
q_{50}	0.00663	x	49

Table 2: Values of parameters used in numerical calculations

Table 2 lists the parameter values used in the numerical calculation. The process of optimization for this case is described as follows: When $PP(n,b,r_g,\pi) = AA - Bn - Gb + F\pi$ and b = 0, the constrained condition (1) for objective function $\max ENPV\left(n,r_g\right)$ becomes $AA - Bn + F\pi > 0$, or $n < \frac{AA + F\pi}{B}$. While it is easily seen from equation (3) that $b \geq 0$ (The integrand and $dPUT_t$ are greater than or equal to 0 and $Y > 0$), and the up boundary limit of *n* for $Max\, ENPV(n,r_g)$ is less or equal to $\frac{AA + F\pi}{B}$, for a given r_g, with $0 \leq r_g \leq \max r_g$, the grid search procedure is employed to locate optimal solutions r_g* and $n*$ that satisfy constraints of $AA - Bn - Gb + F\pi > 0$ and $QD(n,r_g) \leq CON$ on the intervals $\left(0, \frac{AA + F\pi}{B}\right)$ for n and $\left(\min r_g, \max r_g\right)$ for r_g. Since the return rate of investment is a random variable more or less centered at a positive interest rate, we first discuss the cases of $r_g \geq 0$ (in the next section we will discuss the case of $r_g < 0$). Table 3 lists the results of optimization.

n	Optimum r_g	*QD*	$PP(n,b,r_g,\pi)$	$ENPV(n,r_g)$	$b(n,r_g)$
10600	0	0.0218	0.0532	346.3078	5.5670
10800	0	0.0226	0.0527	349.2365	5.9555
11000	0	0.0246	0.0520	353.3249	6.5796
11200	0	0.0260	0.0511	355.3807	6.8648
11400	0	0.0274	0.0505	356.7720	7.0328
11600	0	0.0296	0.0499	358.4064	7.6816
11800	0	0.0321	0.0490	359.2271	8.4012
12000*	0*	0.0351*	0.0485*	359.4955*	10.4334*
12200	0	0.0388	0.0476	359.0794	11.7744
12400	0	0.0425	0.0464	358.4161	13.3046
12600	0	0.0490	0.0447	355.4974	15.7414
12800	0	0.0557	0.0441	349.2164	18.2843

Table 3: Results of optimization

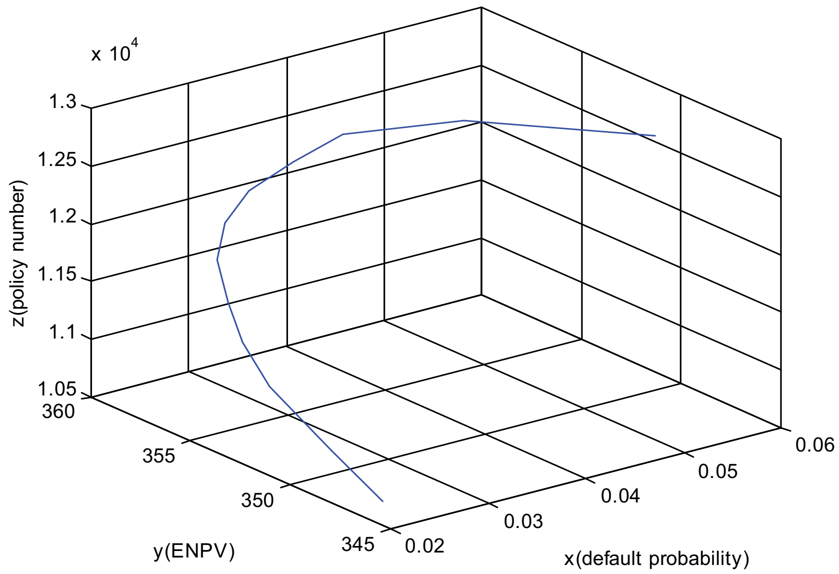

Figure 1: Results of optimization

From Table 3 we gather that optimum solutions are

$$r_g{*} = 0, n{*} = 12000, QD{*}(n, r_g) = 0.0351, PP{*}(n, b(n, r_g), \pi(n, r_g)) = 0.0485,$$
$$b{*}(n, r_g) = 10.4334.$$

Using the equation (5), the optimal boundary of default,

$$BARR_t{*} = {}_tq_x n{*} e^{r_g{*}t} \frac{1+p}{1-h(1+p)} = \frac{1+0.2}{1-0.6(1+0.2)} \cdot 12000\,{}_tq_{49} = 51428.57\,{}_tq_{49},$$

for $0 < t \leq T$. Figure 1 describes the relationship among policy number (*z*-axis in Figure1), expected net present value (*y*-axis in Figure 1) and default probability (*x*-axis in Figure 1). The surface in Figure 1 is a convex quadratic three-dimensional surface and the top of this surface is flatter, which means that the optimal solution of $ENPV$ is insensitive to the change of *n*. From Table 1 we see that when n changes from 11600 to 12000 (12000 to 12400), the change of the values of $ENPV$ is only 1.0891 (1.0854) and the elasticity coefficient is only 0.09 (0.09058). This characteristic is helpful is saving calculation time in optimization, because we can take larger iterative steps without losing accuracy in the process of searching for local optimal solutions.

It should be noted that if the insolvency risk is not reflected in prices, the insolvency risk will increase the net values of insurers, which may be due to two situations where this would be the case. The first is the policyholder’s lack of information regarding the risk-ness of the insurer or inability to assess that risk. The second is when the policy-holder is covered by a credible outside guarantee and is therefore indifferent with regard to the risk of the insurer (see Babel et al., 2005).

If insolvency risk is not considered in the function of prices, simply let *G* = 0 in the equation (12), or $PP(n, b(n, r_g), \pi(n, r_g)) = AA - Bn + F\pi$. With the help of Monte Carlo simulation, we obtain optimal solutions that are listed in Table 4.

$\kappa = 0.24, r_0 = 0.07, \mu = 0.05, \sigma = 0.12$				
σ_1	0.05		0.17	
	Considering insolvency risk in price function	Without considering insolvency risk in price function	Considering insolvency risk in price function	Without considering insolvency risk in price function
$\max ENPV\left(n^*, r_g^*\right)$	440.0831	463.4242	359.4955	395.1984
n^*	17600	17600	12000	12000
$PP(n^*, b(n^*, r_g^*), \pi(n^*, r_g^*))$	0.0398	0.0411	0.0485	0.0527
$\pi(n^*, r_g^*)$	0.0125	0.0128	0.0162	0.0175
Claim payment per unit price after consideration for cost of default risk	0.3138	0.3108	0.3331	0.3308
$b(n^*, r_g^*)$	3.1791	10.9077	10.4330	13.8226
Liability Values	260.8209	259.4283	222.847	238.1774
$QD(n^*, r_g^*)$	0.0162	0.0421	0.0351	0.0425
r_g^*	0	0.06	0	0.06

Table 4: Optimization results using Monte Carlo simulation

From Table 4 we see in contrast with the optimal solutions where the insolvency risk is reflected in prices ($G \neq 0$), the net value of insurer increases at cost of increasing default risks and decreasing claim payment per each unit of price (after consideration for cost of credit risk) although optimal minimum guaranteed return rate increases.

3.3. Convergence of default put option values and default probabilities

For optimization, the convergence of default option values b is very important. By iterative approximation calculation (For iterative equation, please see equation (16)), we find that the iterative values of b are convergent, and the approximating value of b and QD can be found out by limited times of iteration. The iterative process is as follow:

Put the initial value of b_0 into the right side of equation (10), and calculate the value of b_1. Put the value of b_1 into the right side of equation (10), and calculate the value of b_2. Through limited number of cycles, we can get the approximated solution.

Figures 2 and 3 illustrate convergence of the value of default option b and default probability QD. We see from these figures that the iterative process is stable.

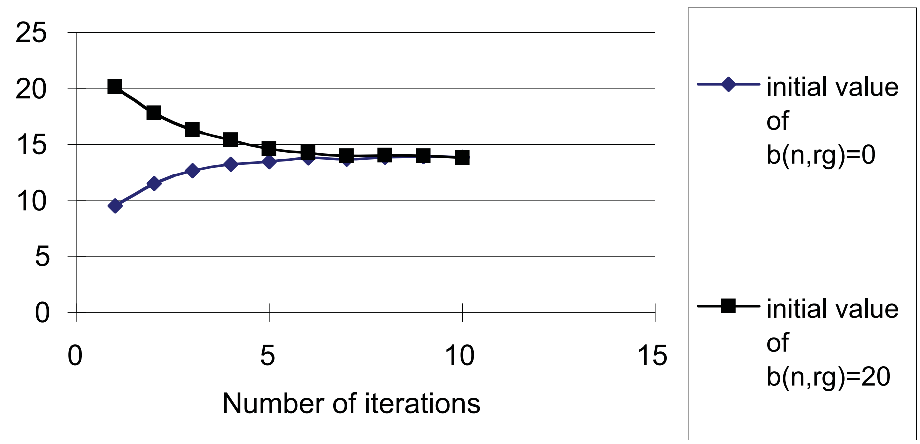

Figure 2: Convergence of iterative process in default put value calculation

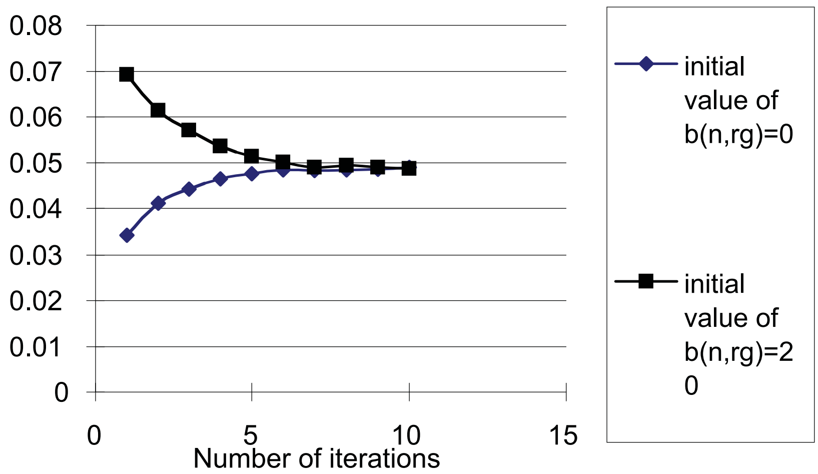

Figure 3: Convergence of iterative process in default probability calculation

From Figure 2 and Figure 3 we can also see that the results of b and QD are insensitive to the initial values of b. The convergent speeds are all very high and the convergence processes are all stable regardless of initial values of b.

4 Discussions and Analysis

4.1. Discussions on the influence of values of minimum guaranteed return rate on risks and benefits of insurers and the insured

By general principles of game theory, insurers should establish their strategy assuming their customers would maximize value to themselves. In this section, we will discuss the selection strategies for insurers based on the ideas of game theory. Because the return rate of investment can and does become negative often, it is naturally necessary to extend our discussion to the cases of minimum guaranteed return rate $r_g < 0$. Table 3 lists the numerical results to demonstrate the influence of values of minimum guarantee return rate r_g on the benefits of insurers and the insured. We set r_g in five different levels (-0.06, -0.03, 0, 0.03, 0.06) to see the corresponding changes in the values of put option, the probability of default, expected net present values, claim payment for unit price (π / PP).

r_g	Case1 (-0.06)	Case2 (-0.03)	Case 3 (0.0)	Case4 (0.03)	Case5 (0.06)
$MaxENPV(n^*, r_g^*)$	390.1872	374.1127	359.495	342.336	324.0123
n^*	13000	12200	12000	11300	10200
$PP(n^*, b(n^*, r_g^*), \pi(n^*, r_g^*))$	0.0482	0.0493	0.0485	0.04880	0.05080
$\pi(n^*, r_g^*)$	0.0156	0.0160	0.01620	0.01630	0.01700
Claim payment for unit price taking off cost of default risk	0.3232	0.3240	0.33310	0.33280	0.32930
$b(n^*, r_g^*)$	5.7894	7.1724	10.4330	13.2914	13.8644
Values of Liability	236.9348	227.7857	222.847	207.736	194.2156
$QD(n^*, r_g^*)$	0.0202	0.0240	0.0351	0.0451	0.0499

Table 5: Influence of values of minimum guarantee return rate on risks and benefits of insurers and the insured

From Table 5 we see that when minimum guaranteed return rate r_g increases, the probability of default also increases, at the same time the expected net present value decreases and claim payment for unit price (taking off cost of credit risk) at first increases then decreases and reaches largest value at the point of $r_g = 0$. From the insurer's perspective, smallest negative minimum guaranteed return rate is best selection based on the criterion of maximizing profit, but it is noticeable that the criterion of maximizing profit is established on the basis of the demand of customers. If the negative minimum guaranteed return decreases the demand of customers, it will result the decrease of insurer's profit. Therefore, the insurer generally will not take smallest negative minimum guaranteed return rate as best choice due to fear of loss of customers. From the consumer's perspective, r_g = 0 is the best selection based on financial aspects. However, minimum return rate guarantee has life insurance protection and that has some value to customers. It is not uncommon for real life customers to accept effective minimum guarantee rate even at the cost of accepting lower claim payment. Insurance companies may consider giving up some value and choosing sub-optimal strategy in order to satisfy customer's need and stimulate demand. It also should be emphasized that minimum guaranteed rate should be a rate at which the company is able to fulfill its obligations. Otherwise, default option may expire in the money. The reality of this threat has been illustrated in the case of Nissan Mutual in Japan, a company that was unable to meet the interest rate guarantee.

4.2. Analysis of effect on when parameters of CON change

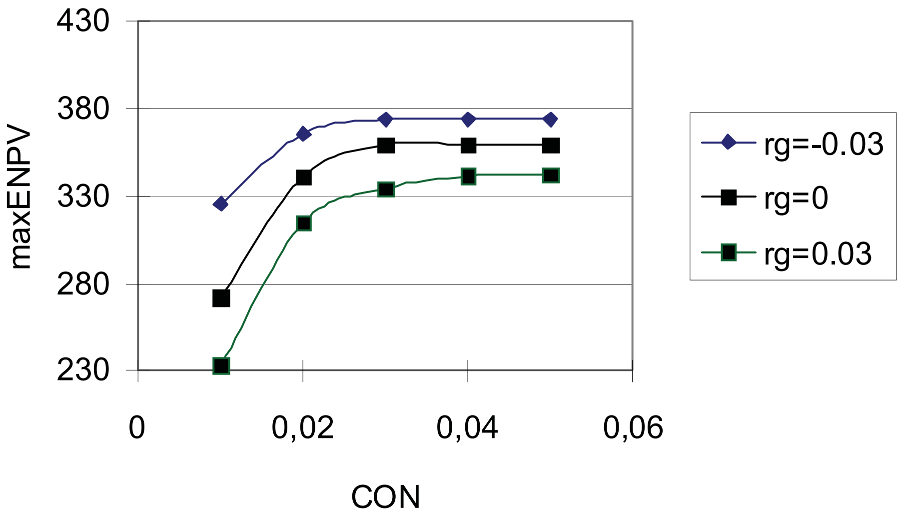

Figure 4: Effect on *ENPV* when parameters of *CON* change

Figure 4 shows the effect of changes of the constraint value of default probability *CON* on the expected net present value of profit. Note that lower value of *CON* decreases the benefit of insurers but provides better security against losses. When constraint value of default probability changes from 0.02 to 0.01, the benefit obtained by insurers decreases sharply especially in the cases of r_g taking larger values. For example, in the case of $r_g = 0.03$, cutting the value of *CON* from 0.02 to 0.01, the value of optimal *ENPV* decreases 26%, but in the case that cutting the value of *CON* from 0.03 to 0.02, the optimal value of *ENPV* only decreases 5.89%. Therefore, the selection of a suitable constraint value of default probability should take into account of marginal effects of credit risk on the value of insurers, but same important thing is that the requirement of regulation and the attitude to risks of the two parties of insurance contracts should be considered.

5 Conclusions

This foregoing analysis discusses measurement of default risk and valuation of equity-linked life insurance contract. Models of default option value and default probability is established incorporating stochastic interest rates, supply and demand, stochastic investment return, mortality rates. The values of default boundary and price of insurance policy are determined by maximizing expected net present value of profit subject to the constrains of price >0 and default probability $\leq$ a constant with the help of stochastic optimization and Monte Carlo simulation. The term structure of stochastic rates of return on the asset portfolo is based on the Cox, Ingersoll, Ross (1985) model. Convergence of iterative process of finding approximating value of default option and the effects of minimum guarantee return rate on default probability, price and expected net present value of profit are discussed. Our results indicate that the convergence speeds are very high and that convergence processes are all very stable no matter what initial values of b are taken in feasible solution area of n and b. The results also indicate that the default probability increases with the increase of minimum guaranteed return rate r_g. Increasing minimum guaranteed return rate will decrease optimal expected net present value and change claim payment. Therefore, risk and benefit, advantage and disadvantage must be weighted and balanced in selecting minimum return rate guarantee. The further work will be focus on discussing to use other credit measurement such as value at risk (VaR) and conditional value at risk (CVaR) as a constraint in establishing optimal model. Finally we could also extend our model to the case of level premium insurance policies.

Appendix

Proof of continuity of equation (3):

The equation (3) can be written as

$$\begin{aligned} b(n,r_g) = E^Q\left(\int_0^Y A(t)e^{-B(t)r}\, d\max\left(X_t - D_t, 0\right)\right) = \\ = \int_0^Y A(t)e^{-B(t)r} dE^Q\left(\max\left(X_t - D_t, 0\right)\right). \end{aligned} \tag{20}$$

where

$$X_t = \left(\max\left(D_t h, GUE_t\right)\right)(1+p) = \left(D_t h + \max\left(GUE_t - D_t h, 0\right)\right)(1+p).$$

From equation (20), we can easily know that the first term of the integrand of b---- $A(t)e^{-B(t)r}$ is a continuous function, where

$$A(t)=\left(\frac{2\gamma e^{(\kappa+\gamma)t/2}}{g(t)}\right)^{2\kappa\mu/\sigma^2}$$

$$B(t)=\frac{2\left(e^{\gamma t}-1\right)}{g(t)}$$

$$g(t)=2\gamma+(\kappa+\gamma)\left(e^{\gamma t}-1\right)$$

$$\gamma=\sqrt{\kappa^2+2\sigma^2}$$

Let $R=X_t-D_t$, then $dE^Q\left(\max\left(X_t-D_t,0\right)\right)=\frac{dE^Q\left(\max\left(R,0\right)\right)}{dt}dt$.

When $R\geq 0$, $dE^Q\max\left(R,0\right)=\frac{d}{dt}E^Q\left(X_t\right)dt-\frac{dE^Q\left(D_t\right)}{dt}dt$

$$=\frac{d}{dt}E^Q\left(\max\left(D_th,GUE_t\right)(1+p)\right)-\frac{dE^Q\left(D_t\right)}{dt}dt$$

$$=\begin{cases}\frac{d}{dt}E^Q\left(D_th(1+p)\right)-\frac{dE^Q\left(D_t\right)}{dt}dt & \text{when } D_th>GUE_t,\\ =\left(h(1+p)-1\right)\frac{dE^Q\left(D_t\right)}{dt}dt & \\ \frac{d}{dt}E^Q\left(GUE_t(1+p)\right)-\frac{dE^Q\left(D_t\right)}{dt}dt & \text{otherwise.}\\ =f_x(t)e^{r_gt}(1+p)n-\frac{dE^Q\left(D_t\right)}{dt}dt & \end{cases}$$

Based on the Cauchy-Schwarz inequality,

$$\frac{dE^Q\left(D_t\right)}{dt}dt=E^Q\left(rD_tdt+\sigma_1D_t\frac{dw^1}{dt}dt\right)\leq E^Q\left(rD_tdt\right)+\sqrt{E^Q\left(\sigma_1D_t\right)^2}\sqrt{\frac{d}{dt}E^Q\left(w^1\right)^2dt}$$

Since the variance of standard Wiener process $D^Q(w^1)=t$, the expectation of standard Wiener process $E^Q(w^1)=0$ and $E^Q(w^1)^2=D^Q(w^1)-\left(E^Q(w^1)\right)^2=t$,

$$\frac{dE^Q(D_t)}{dt}dt\leq E^Q(rD_t)dt+\sqrt{E^Q(\sigma_1D_t)^2}\sqrt{dt}\geq 0$$

so

$$dE^Q \max(R,0) \le \begin{cases} (h(1+p)-1)\left(E^Q(rD_t)dt + \sqrt{E^Q(\sigma_1 D_t)^2}\sqrt{dt}\right) & \text{when } D_t h > GUE_t \\ f_x(t)e^{r_g t}(1+p)n & \text{otherwise.} \end{cases}$$

When $R < 0$, $dE^Q \max(R,0) = 0$. Therefore $E^Q \max(R,0)$ is differentiable and continuous. And since $A(t)e^{-B(t)r}$ is also continuous, $b(n, r_g)$ is a continuous function. Similarly, we can prove that $\pi^1(n, x, r_g, Y)$ is also a continuous function.

While the price of life insurance contract and the expected net present value are the functions of $b(n, r_g)$ and $\pi(n, x, r_g, Y)$, $ENPV(n, r_g)$ and $PP(n, b(n, r_g), \pi(n, r_g))$ are also continuous functions.

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